Laplace Transform Properties (3A)

Young Won Lim
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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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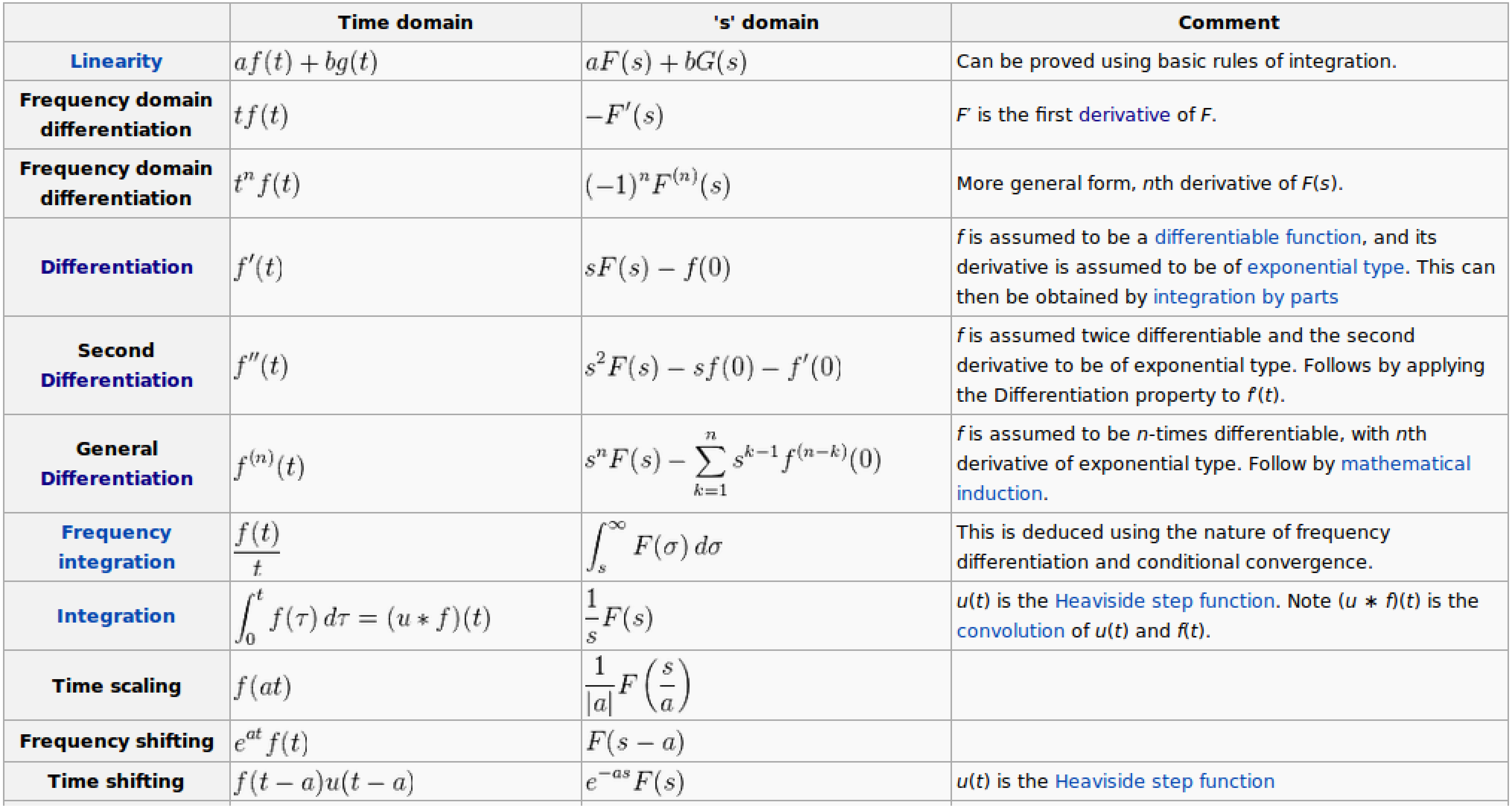
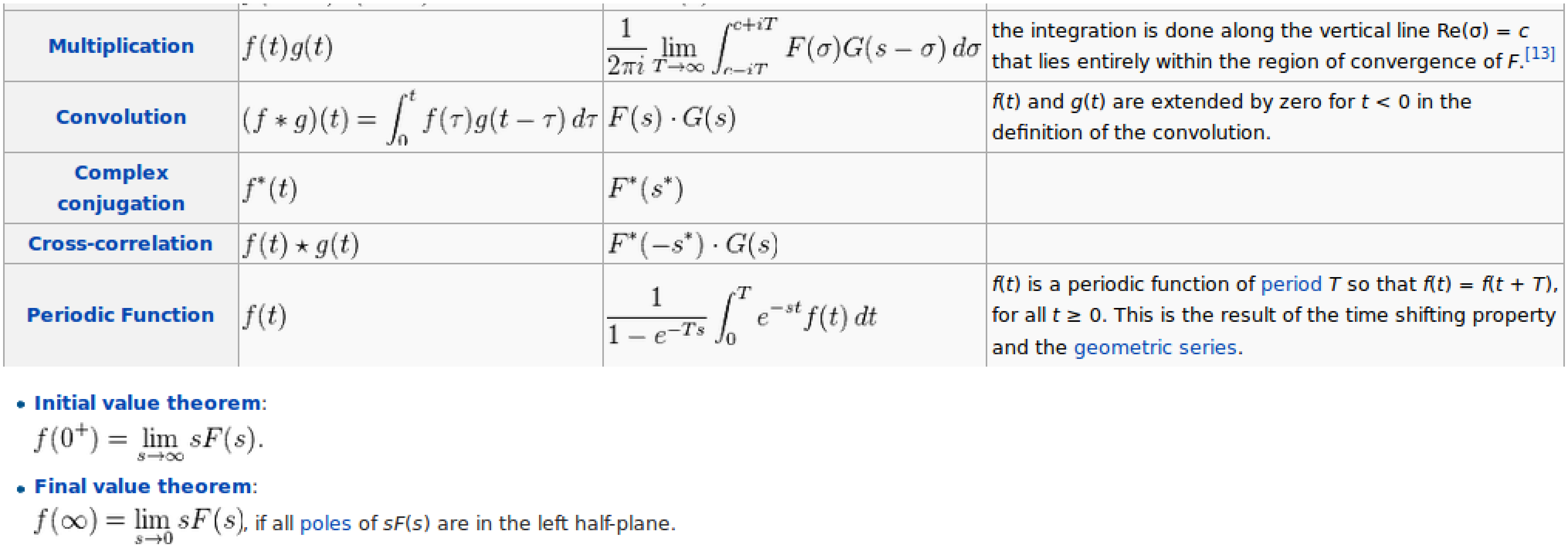
Laplace Transform Properties (1)



	Time domain	's' domain	Comment
Linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$	Can be proved using basic rules of integration.
Frequency domain differentiation	$tf(t)$	$-F'(s)$	F' is the first derivative of F.
Frequency domain differentiation	$t^n f(t)$	$(-1)^n F^{(n)}(s)$	More general form, nth derivative of $F(s)$.
Differentiation	$f'(t)$	$sF(s) - f(0)$	f is assumed to be a differentiable function, and its derivative is assumed to be of exponential type. This can then be obtained by integration by parts
Second Differentiation	$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	f is assumed twice differentiable and the second derivative to be of exponential type. Follows by applying the Differentiation property to $f'(t)$.
General Differentiation	$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0)$	f is assumed to be n-times differentiable, with nth derivative of exponential type. Follow by mathematical induction.
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)\,d\sigma$	This is deduced using the nature of frequency differentiation and conditional convergence.
Integration	$\int_0^t f(\tau)\,d\tau = (u * f)(t)$	$\frac{1}{s}F(s)$	$u(t)$ is the Heaviside step function. Note $(u * f)(t)$ is the convolution of $u(t)$ and $f(t)$.
Time scaling	$f(at)$	$\frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right)$	
Frequency shifting	$e^{at}f(t)$	$F(s-a)$	
Time shifting	$f(t-a)u(t-a)$	$e^{-as}F(s)$	$u(t)$ is the Heaviside step function

http://en.wikipedia.org/wiki/Laplace_transform

Laplace Transform Properties (2)



Multiplication	$f(t)g(t)$	$\frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} F(\sigma)G(s-\sigma)\,d\sigma$	the integration is done along the vertical line Re(σ) = *c* that lies entirely within the region of convergence of *F*.[13]
Convolution	$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)\cdot G(s)$	*f*(*t*) and *g*(*t*) are extended by zero for *t* < 0 in the definition of the convolution.
Complex conjugation	$f^*(t)$	$F^*(s^*)$	
Cross-correlation	$f(t)\star g(t)$	$F^*(-s^*)\cdot G(s)$	
Periodic Function	$f(t)$	$\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)\,dt$	*f*(*t*) is a periodic function of period *T* so that *f*(*t*) = *f*(*t* + *T*), for all *t* ≥ 0. This is the result of the time shifting property and the geometric series.

- **Initial value theorem**:

$$f(0^+)=\lim_{s\to\infty} sF(s).$$

- **Final value theorem**:

$f(\infty)=\lim_{s\to 0} sF(s)$, if all poles of $sF(s)$ are in the left half-plane.

http://en.wikipedia.org/wiki/Laplace_transform

Differentiation in the s-domain (1)

$$f(t) \quad \Longleftrightarrow \quad F(s) \qquad F(s) = \int_0^\infty f(t)\cdot e^{-st}\,dt$$

$$-t\,f(t) \quad \Longleftrightarrow \quad F'(s) \qquad \frac{d}{ds}F(s) = \int_0^\infty \frac{\partial}{\partial s}\left[f(t)\cdot e^{-st}\right]dt = \int_0^\infty (-t)\,f(t)\cdot e^{-st}\,dt$$

$$+t^2 f(t) \quad \Longleftrightarrow \quad F''(s) \qquad \frac{d^2}{ds^2}F(s) = \int_0^\infty \frac{\partial^2}{\partial s^2}\left[f(t)\cdot e^{-st}\right]dt = \int_0^\infty (-t)^2 f(t)\cdot e^{-st}\,dt$$

$$-t^3 f(t) \quad \Longleftrightarrow \quad F^{(3)}(s) \qquad \frac{d^3}{ds^3}F(s) = \int_0^\infty \frac{\partial^3}{\partial s^3}\left[f(t)\cdot e^{-st}\right]dt = \int_0^\infty (-t)^3 f(t)\cdot e^{-st}\,dt$$

$$t^n f(t) \quad \Longleftrightarrow \quad (-1)^n \frac{d^n}{ds^n}F(s)$$

Differentiation in the s-domain(2)

$$1 \Longleftrightarrow \frac{1}{s}$$

$$-t \Longleftrightarrow -\frac{1}{s^2} = \frac{d}{ds}\left(\frac{1}{s}\right)$$

$$t^2 \Longleftrightarrow \frac{2}{s^3} = \frac{d}{ds}\left(-\frac{1}{s^2}\right)$$

$$-t^3 \Longleftrightarrow -\frac{6}{s^4} = \frac{d}{ds}\left(\frac{2}{s^3}\right)$$

$$t^n \Longleftrightarrow \frac{n!}{s^{n+1}}$$

Differentiation in the t-domain (1)

$$f(t) \Longleftrightarrow F(s) \qquad F(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt$$

$$\int_0^\infty f'(t) \cdot e^{-st}\, dt = \Big[f(t) \cdot e^{-st}\Big]_0^\infty - \int_0^\infty (-s) f(t) \cdot e^{-st}\, dt$$

$$= -f(0) + s \int_0^\infty f(t) \cdot e^{-st}\, dt = sF(s) - f(0)$$

$$f'(t) \Longleftrightarrow \underbrace{sF(s) - f(0)}_{}$$

$$f''(t) \Longleftrightarrow \underbrace{s\big(sF(s) - f(0)\big) - f'(0)}_{}$$

$$f^{(3)}(t) \Longleftrightarrow \underbrace{s\Big(s\big(sF(s) - f(0)\big) - f'(0)\Big) - f''(0)}_{}$$

Differentiation in the t-domain (2)

$$f(t) \Longleftrightarrow F(s)$$

$$f'(t) \Longleftrightarrow sF(s) - f(0)$$

$$f''(t) \Longleftrightarrow s^2F(s) - sf(0) - f'(0)$$

$$f^{(3)}(t) \Longleftrightarrow s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$f^{(n)}(t) \Longleftrightarrow s^nF(s) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - \cdots - f^{(n-1)}(0)$$

Differentiation in the t-domain (3)

$$f^{(n)}(t) \Longleftrightarrow s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s^1 f^{(n-2)}(0) - f^{(n-1)}(0)$$

s^n $F(s)$	s^{n-1} $f^{(0)}(0)$	s^{n-2} $f^{(1)}(0)$	s^1 $f^{(n-2)}(0)$	s^0 $f^{(n-1)}(0)$
	$n-1+0$	$n-2+1$	$1+n-2$	$0+n-1$

$$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}$$

$$s(\cdots s(s(sF(s) - f(0)) - f'(0)) - f''(0) \cdots) - f^{(n-1)}(0)$$

Differentiation Properties

$$\frac{d^n}{dt^n} f(t) \quad \Longleftrightarrow \quad s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \ \cdots \ - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

$$t^n f(t) \quad \Longleftrightarrow \quad (-1)^n \cdot \frac{d^n}{ds^n} F(s)$$

Integration in the t-domain

$$f(t) \longleftrightarrow F(s)$$

$$\int_0^t f(\tau)\,d\tau \longleftrightarrow \frac{F(s)}{s}$$

$$\begin{tikzcd}
\int_0^t f(\tau)\,d\tau \arrow[r, leftrightarrow] \arrow[d, equal] & ? \arrow[d, equal] \\
g(t) \arrow[r, leftrightarrow] \arrow[dd, bend right] & G(s) \\
f(t) \arrow[r, leftrightarrow] \arrow[d, equal] & F(s) \arrow[d, equal] \\
g'(t) \arrow[r, leftrightarrow] & sG(s) - g(0)
\end{tikzcd}$$

$$f(t) = \frac{d}{dt}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{d}{dt}g(t)$$

$$g(t) = \int_0^t f(\tau)\,d\tau$$

$$g(0) = \int_0^0 f(\tau)\,d\tau = 0$$

$$G(s) = \frac{F(s)}{s}$$

Unilateral and Bilateral Laplace Transform

Unilateral Laplace Transform

$$F(s) = \int_{0}^{+\infty} f(t) e^{-st}\, dt$$

Including an Impulse at the origin

$$F_{-}(s) = \int_{0^{-}}^{+\infty} f(t) e^{-st}\, dt$$

Excluding an Impulse at the origin

$$F_{+}(s) = \int_{0^{+}}^{+\infty} f(t) e^{-st}\, dt$$

Bilateral Laplace Transform

$$F_{2}(s) = \int_{-\infty}^{+\infty} f(t) e^{-st}\, dt$$

Unilateral and Bilateral Laplace Transform

Unilateral Laplace Transform

Including an Impulse at the origin

$$F_-(s) = \int_{0^-}^{+\infty} f(t) e^{-st}\, dt \qquad f'(t) \Longleftrightarrow sF_-(s) - f(0^-)$$

Excluding an Impulse at the origin

$$F_+(s) = \int_{0^+}^{+\infty} f(t) e^{-st}\, dt \qquad f'(t) \Longleftrightarrow sF_+(s) - f(0^+)$$

Bilateral Laplace Transform

$$F_2(s) = \int_{-\infty}^{+\infty} f(t) e^{-st}\, dt \qquad f'(t) \Longleftrightarrow sF_2(s) - f(0)$$

Translation in the s-domain

$$f(t) \longleftrightarrow F(s) \qquad F(s) = \int_0^\infty f(t)\cdot e^{-st}\,dt$$

$$e^{+at} f(t) \longleftrightarrow F(s-a) \qquad F(s-a) = \int_0^\infty f(t)\cdot e^{-(s-a)t}\,dt = \int_0^\infty \left[e^{+at} f(t)\right] e^{-st}\,dt$$

$$e^{\pm at} f(t) \longleftrightarrow F(s \mp a)$$

Translation in the t-domain

$$f(t) \longleftrightarrow F(s) \qquad F(s) = \int_0^{\infty} f(t) \cdot e^{-st}\, dt$$

$$f(t-a)u(t-a) \longleftrightarrow e^{-as} F(s)$$

$$f(t \mp a)u(t \mp a) \longleftrightarrow e^{\mp as} F(s)$$

$$\int_0^{\infty} f(t-a)u(t-a) \cdot e^{-st}\, dt$$

$$= \int_0^{a} f(t-a)\cancel{u(t-a)} \cdot e^{-st}\, dt + \int_a^{\infty} f(t-a)u(t-a) \cdot e^{-st}\, dt$$

$$= \int_a^{\infty} f(t-a) \cdot e^{-st}\, dt$$

$$\nu = t-a \qquad d\nu = dt$$

$$0 = a-a$$

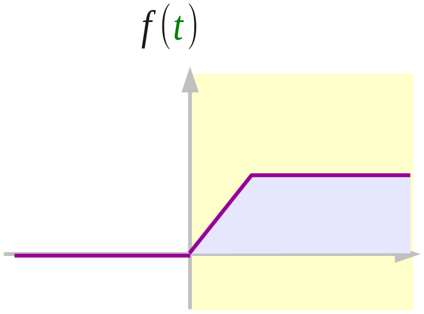
$$= \int_0^{\infty} f(\nu) \cdot e^{-s(\nu+a)}\, d\nu$$

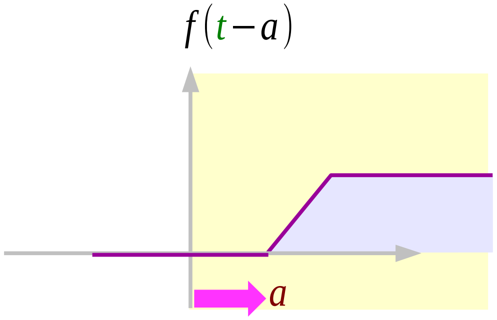
$$= e^{-as} \cdot \int_0^{\infty} f(\nu) \cdot e^{-s\nu}\, d\nu$$

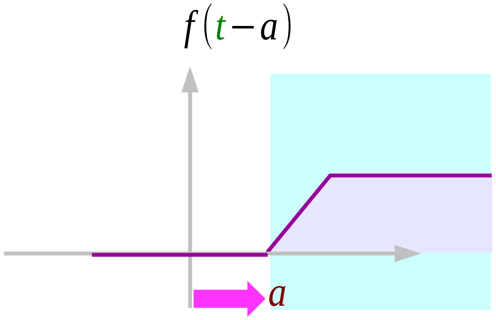
$$= e^{-as} \cdot F(s)$$

shift right : always o.k.
shift left: only when no information is lost during improper integration by the left shift

Shift Right f(t)







$$\int_0^\infty f(t)\cdot e^{-st}\,dt$$

$$\int_0^\infty f(t-a)\cdot e^{-st}\,dt$$

$$= \int_0^a f(t-a)\cdot e^{-st}\,dt$$

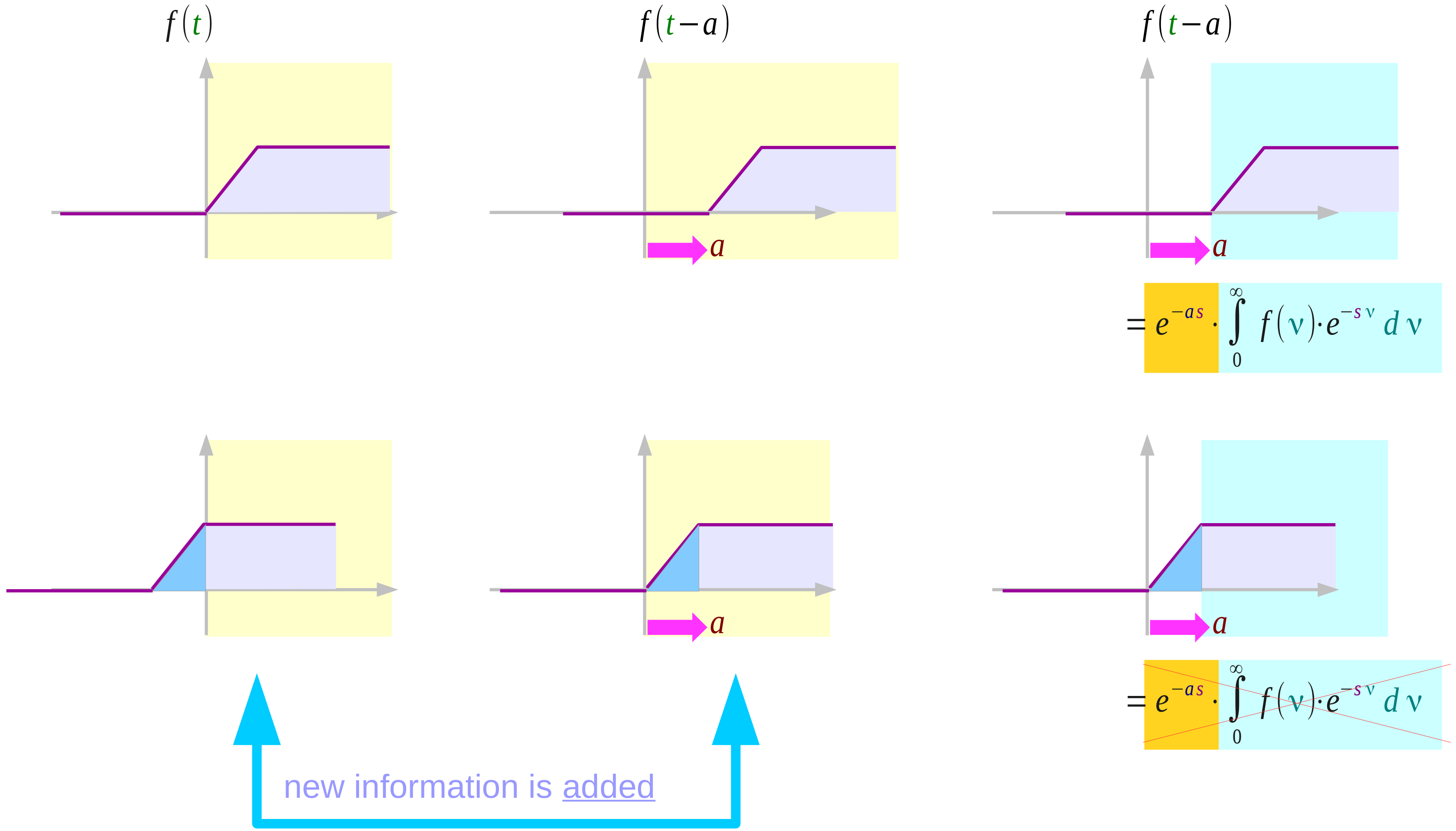
$$+ \boxed{\int_a^\infty f(t-a)e^{-st}\,dt}$$

$$= \boxed{\int_a^\infty f(t-a)e^{-st}\,dt}$$

$$= \int_0^\infty f(\nu)\cdot e^{-s(\nu+a)}\,d\nu$$

$$= e^{-as}\cdot\int_0^\infty f(\nu)\cdot e^{-s\nu}\,d\nu$$

Shift Right f(t)

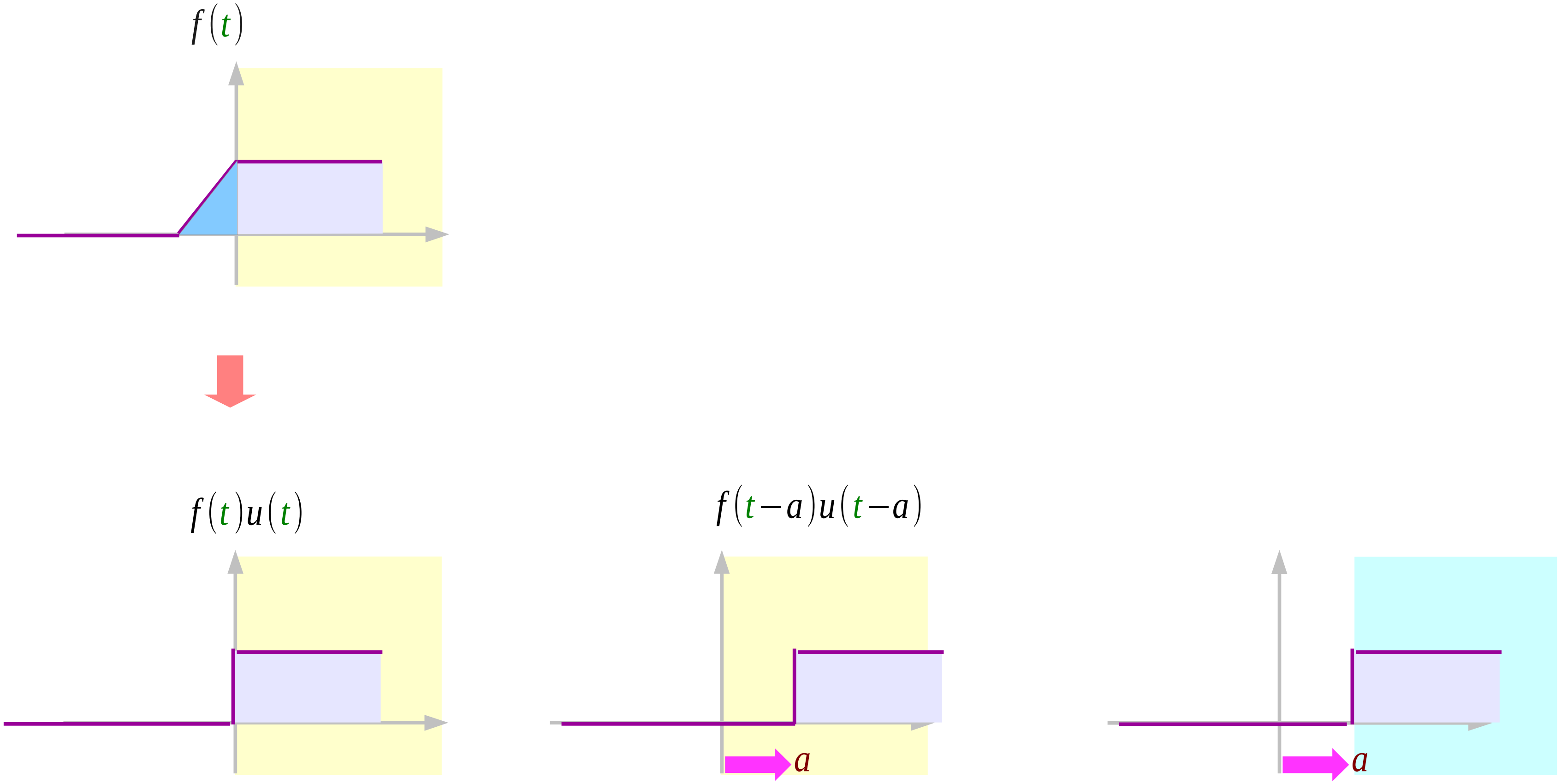


$$= e^{-as} \cdot \int_0^\infty f(\nu)\cdot e^{-s\nu}\, d\nu$$

$$\xcancel{= e^{-as} \cdot \int_0^\infty f(\nu)\cdot e^{-s\nu}\, d\nu}$$

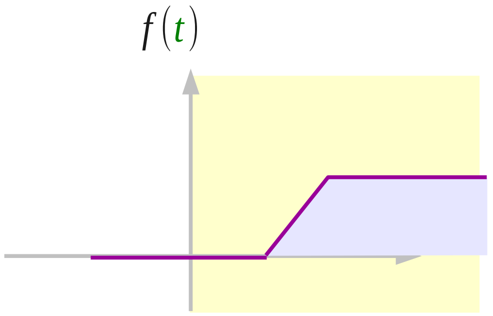
new information is added

Shift Right f(t)u(t)



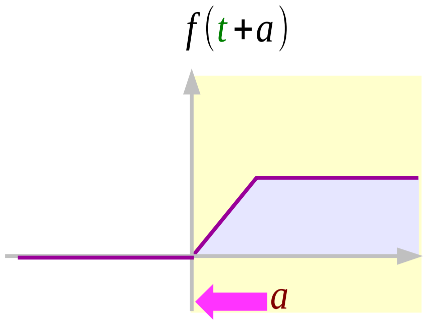
$$= e^{-as} \cdot \int_0^\infty f(\nu) \cdot e^{-s\nu}\, d\nu$$

Shift Left f(t)

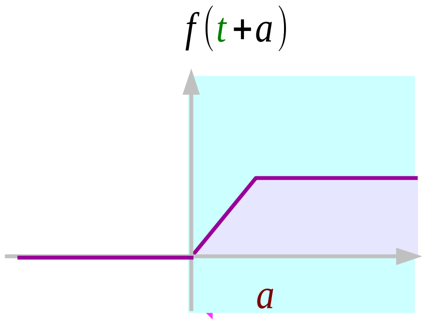


$$\int_0^\infty f(t)\cdot e^{-st}\,dt$$

$$= \int_a^\infty f(t)\cdot e^{-st}\,dt$$



$$\int_0^\infty f(t+a)\cdot e^{-st}\,dt$$

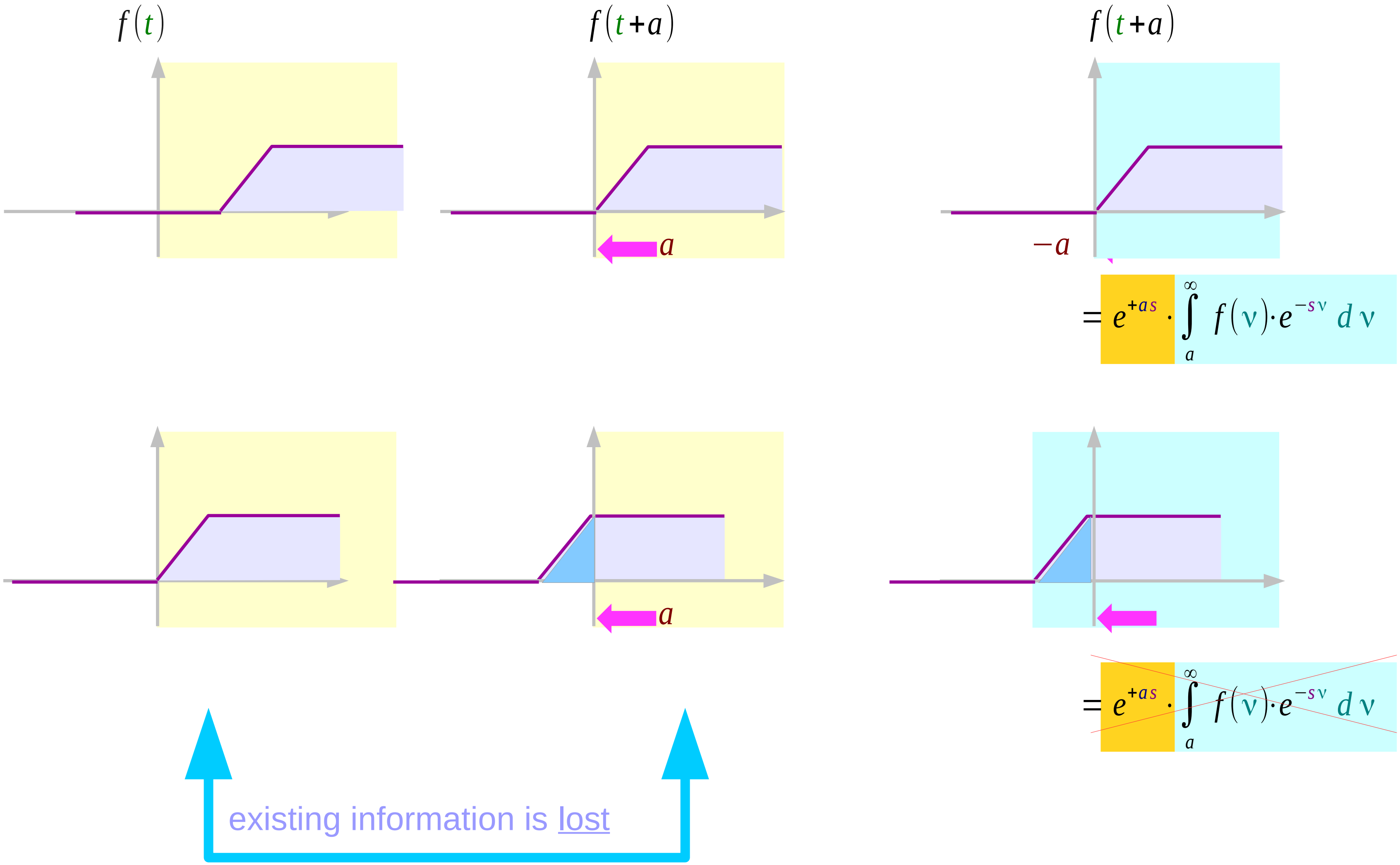


$$= \int_0^\infty f(t+a)e^{-st}\,dt$$

$$= \int_a^\infty f(\nu)\cdot e^{-s(\nu-a)}\,d\nu$$

$$= e^{+as}\cdot\int_a^\infty f(\nu)\cdot e^{-s\nu}\,d\nu$$

Shift Left f(t)



$$= e^{+as} \cdot \int_{a}^{\infty} f(\nu) \cdot e^{-s\nu} \, d\nu$$

$$\cancel{= e^{+as} \cdot \int_{a}^{\infty} f(\nu) \cdot e^{-s\nu} \, d\nu}$$

existing information is lost

Translation Properties

$$e^{\pm at} f(t) \Longleftrightarrow F(s \mp a)$$

$$f(t \mp a) u(t \mp a) \Longleftrightarrow e^{\mp as} F(s)$$

shift right : always o.k.
shift left: only when no information is lost during improper integration by the left shift

Initial Value Theorem

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

$$F(s) = \int_{0^-}^{\infty} f(t) \cdot e^{-st}\, dt$$

$$sF(s) - f(0^-) = \int_{0^-}^{\infty} f'(t) \cdot e^{-st}\, dt$$

$$\lim_{s \to \infty} sF(s)$$

$$= f(0^-) + \lim_{s \to \infty} \int_{0^-}^{\infty} f'(t) \cdot e^{-st}\, dt$$

$$= f(0^-) + f(0^+) - f(0^-)$$

$$= f(0^+)$$

$$\lim_{s \to \infty} \int_{0^-}^{\infty} f'(t) \cdot e^{-st}\, dt$$

$$= \lim_{s \to \infty} \left[\lim_{\epsilon \to 0^+} \int_{0^-}^{\epsilon} f'(t) \cdot e^{-st}\, dt + \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\infty} f'(t) \cdot e^{-st}\, dt \right]$$

$$= \lim_{s \to \infty} \left[\lim_{\epsilon \to 0^+} \int_{0^-}^{\epsilon} f'(t) \cdot 1\, dt + \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\infty} f'(t) \cdot e^{-st}\, dt \right]$$

$$= \left[f(t) \right]_{0^-}^{0^+} + \lim_{\epsilon \to 0^+} \int_{\epsilon}^{\infty} f'(t) \cdot \left(\lim_{s \to \infty} e^{-st} \right) dt$$

$$= f(0^+) - f(0^-)$$

Final Value Theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s)$$

$$F(s) = \int_{0^-}^{\infty} f(t) \cdot e^{-st}\, dt$$

$$s\,F(s) - f(0^-) = \int_{0^-}^{\infty} f'(t) \cdot e^{-st}\, dt$$

$$\lim_{s \to 0} \int_{0^-}^{\infty} f'(t) \cdot e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} f'(t) \cdot \lim_{s \to 0} e^{-st}\, dt$$

$$= \int_{0^-}^{\infty} f'(t)\, dt$$

$$\lim_{s \to 0} \left[s\,F(s) - f(0^-) \right] = \lim_{s \to 0} \int_{0^-}^{\infty} f'(t) \cdot e^{-st}\, dt$$

$$\lim_{s \to \infty} s\,F(s) = f(0^-) + \lim_{s \to 0} \int_{0^-}^{\infty} f'(t) \cdot e^{-st}\, dt$$

$$= f(0^-) + f(\infty) - f(0^-)$$

$$= f(\infty)$$

Laplace Transform of Convolution Integrals

$$f(t) \quad \Longleftrightarrow \quad F(s)$$

$$g(t) \quad \Longleftrightarrow \quad G(s)$$

$$f(t) * g(t) \quad \Longleftrightarrow \quad F(s)G(s)$$

$$\int_0^t f(t-\tau)g(\tau)\,d\tau$$

$$\int_0^t f(\tau)g(t-\tau)\,d\tau$$

Laplace Transform of Convolution Integrals

$$F(s)G(s) = \left[\int_0^\infty e^{-s\tau} f(\tau)\,d\tau\right]\left[\int_0^\infty e^{-s\beta} g(\beta)\,d\beta\right]$$

$$= \int_0^\infty \left[\int_0^\infty e^{-s(\tau+\beta)} f(\tau)\,d\tau\right] g(\beta)\,d\beta$$

$$\tau+\beta = t \quad d\tau = dt$$
$$\tau = 0 \quad \rightarrow \quad t = \beta$$

$$= \int_0^\infty \left[\int_\beta^\infty f(t-\beta)e^{-st}\,dt\right] g(\beta)\,d\beta$$

$$= \int_0^\infty \left[\int_\beta^\infty f(t-\beta)g(\beta)e^{-st}\,dt\right] d\beta$$

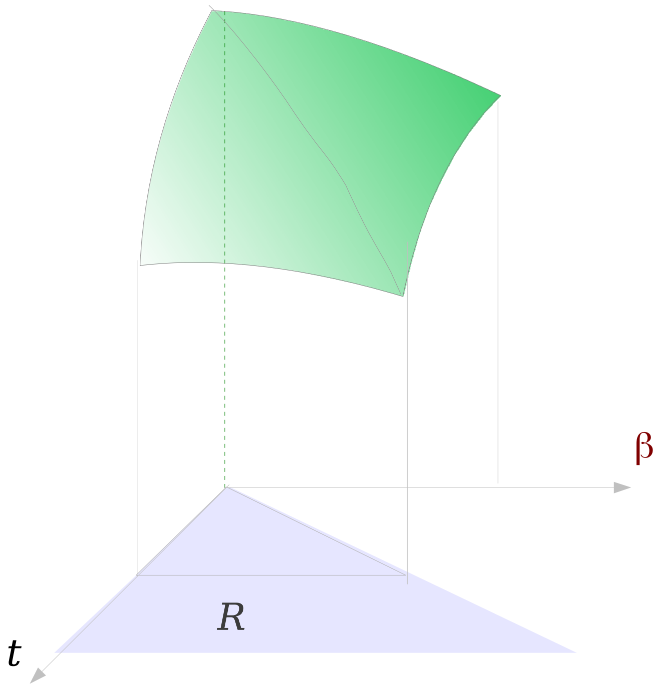
$$\int_0^\infty \int_\beta^\infty dt\,d\beta$$

$$= \int_0^\infty \left[\int_0^t f(t-\beta)g(\beta)e^{-st}\,d\beta\right] dt$$

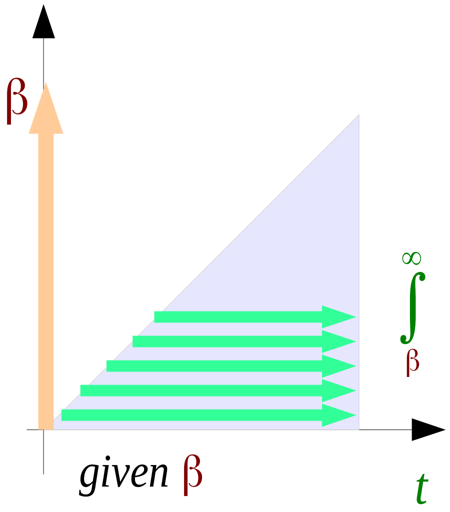
$$\int_0^\infty \int_0^t d\beta\,dt$$

$$= \int_0^\infty \left[\int_0^t f(t-\beta)g(\beta)\,d\beta\right] e^{-st}\,dt$$

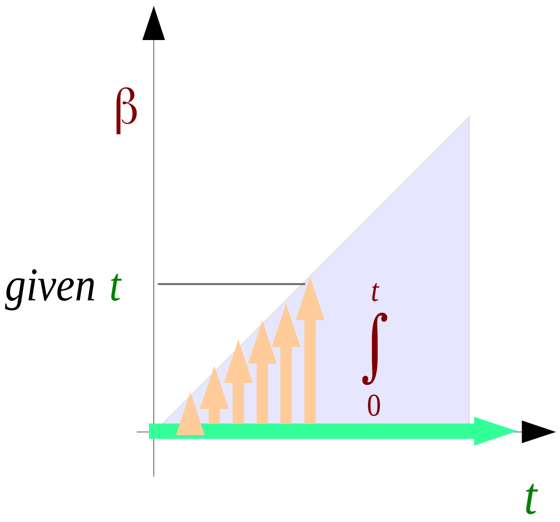
$$h(t,\beta) = f(t-\beta)g(\beta)e^{-st}$$



Laplace Transform of Convolution Integrals



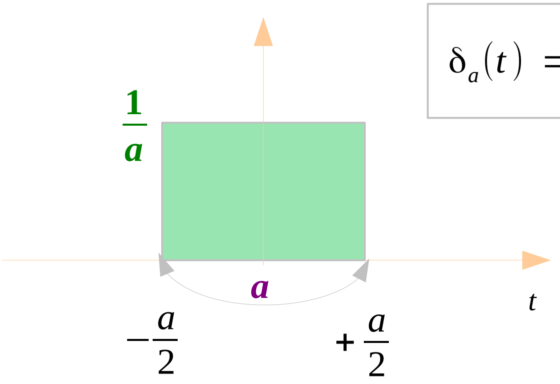
$$\int_0^\infty \left[\int_\beta^\infty f(t-\beta)\, g(\beta)\, e^{-st}\, dt \right] d\beta$$



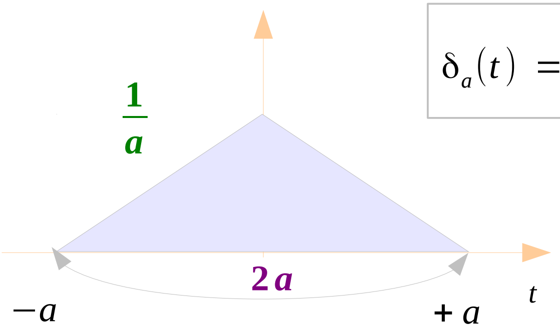
$$\int_0^\infty \left[\int_0^t f(t-\beta)\, g(\beta)\, e^{-st}\, d\beta \right] dt$$

Matthew

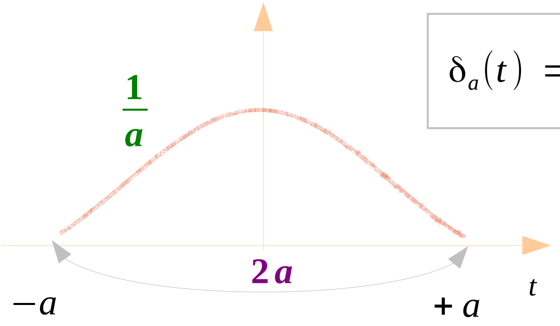
The Unit Impulse



$$\delta_a(t) = \frac{1}{a}\,\boldsymbol{rect}\left(\frac{1}{a}\cdot t\right)$$



$$\delta_a(t) = \frac{1}{a}\left(1 - \frac{1}{a}|t|\right)$$



$$\delta_a(t) = \frac{1}{a}\,\mathbf{exp}\left(\frac{\pi}{a^2}\cdot t^2\right)$$

$$\lim_{a\to 0}\delta_a(t) = \delta(t)$$

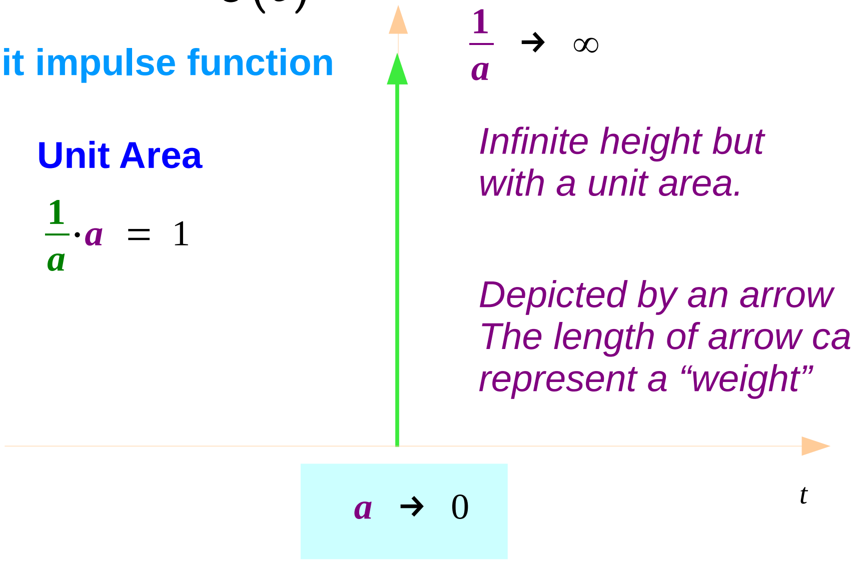
The shape does not matter in the limit
But the area matters : The Unit Area

Dirac Delta $\boldsymbol{\delta(t)}$

Unit impulse function

Unit Area

$$\frac{1}{a}\cdot a = 1$$



$\frac{1}{a} \to \infty$

Infinite height but
with a unit area.

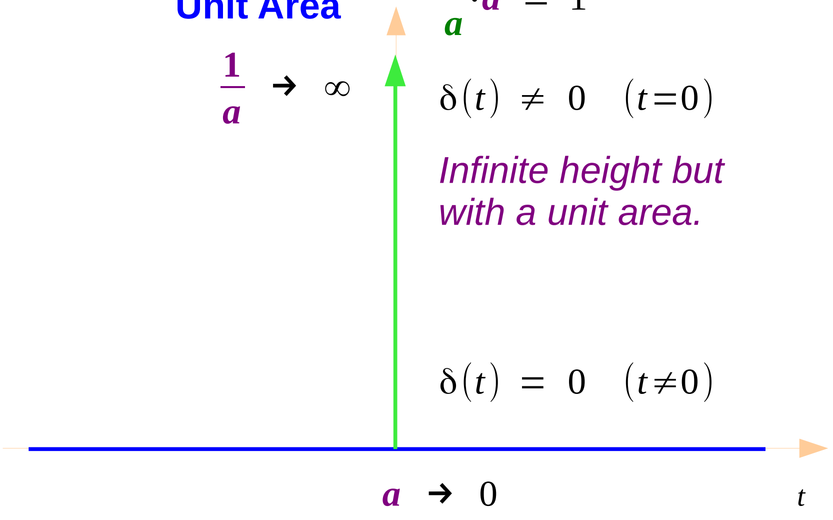
Depicted by an arrow
The length of arrow can
represent a “weight”

$a \to 0$

The Properties of the Delta Function

$\delta(t)$

Unit Area



$\frac{1}{a} \cdot a = 1$

$\delta(t) \neq 0 \quad (t = 0)$

Infinite height but with a unit area.

$\delta(t) = 0 \quad (t \neq 0)$

The Equivalence Property

$$g(t)\,\delta(t) = g(0)\,\delta(t)$$

$$g(t)\,\delta(t-t_0) = g(t_0)\,\delta(t-t_0)$$

The Sampling Property

$$\int_{-\infty}^{+\infty} g(t)\,\delta(t)\,dt = g(0)$$

$$\int_{-\infty}^{+\infty} g(t)\,\delta(t-t_0)\,dt = g(t_0)$$

An Even Function

$$\delta(-t) = \delta(t)$$

The Replication Property

$$\int_{-\infty}^{+\infty} g(\tau)\,\delta(t-\tau)\,d\tau = g(t)$$

The Replication Property

$$g(t_0) = \int_{-\infty}^{+\infty} g(t)\delta(t-t_0)\, dt$$

$t \leftarrow t_0$ $\qquad$ $\tau \leftarrow t$

$$g(t) = \int_{-\infty}^{+\infty} g(t)\delta(\tau-t)\, d\tau$$

$\delta(-t) = \delta(t)$

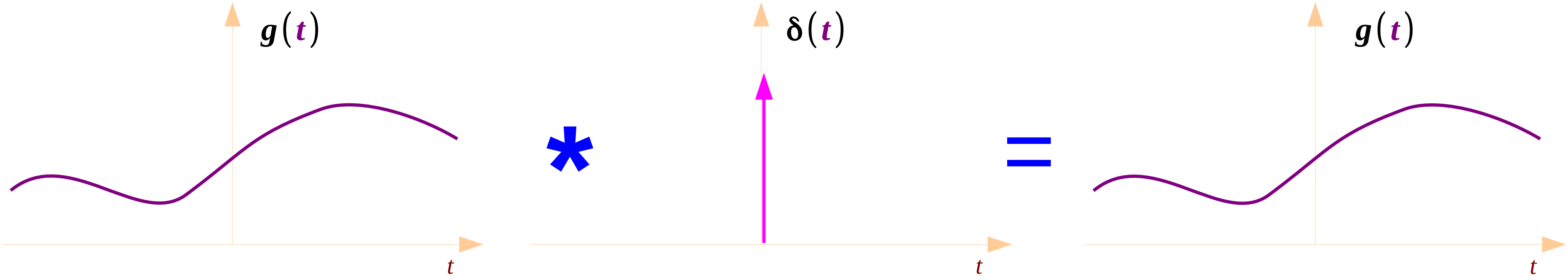
$$g(t) = \int_{-\infty}^{+\infty} g(t)\delta(t-\tau)\, d\tau$$

Impulse response

$$\delta(t) \Rightarrow \boxed{g(t)} \Rightarrow g(t)$$

Replication

$$g(t) \Rightarrow \boxed{\delta(t)} \Rightarrow g(t)$$



Partial Fraction Methods

$$F(s) = \int_{0}^{+\infty} f(t) e^{-st} \, dt$$

$$\int_{0}^{+\infty} \delta(t) e^{-st} \, dt \quad = \int_{0}^{+\infty} \delta(t) e^{-s \cdot 0} \, dt = 1$$

$$\begin{tikzcd}
\delta(t) \arrow[d, bend right, "\frac{d}{dt}"'] \arrow[r, leftrightarrow] & 1 \\
u(t) \arrow[d, bend right, "\frac{d}{dt}"'] \arrow[r, leftrightarrow] & \frac{1}{s} \\
t\,u(t) \arrow[r, leftrightarrow] & \frac{1}{s^2}
\end{tikzcd}$$

Periodic Functions

$$f(t) = f(t+T)$$

piecewise continuous

periodic (period: T)

$$f(t) \Longleftrightarrow \frac{1}{1-e^{-Ts}} \int_0^T f(t) e^{-st}\, dt$$

$$F(s) = \int_0^{+\infty} f(t) e^{-st}\, dt \quad = \int_0^T f(t) e^{-st}\, dt + \int_T^{+\infty} f(t) e^{-st}\, dt$$

$$\int_T^{+\infty} f(t) e^{-st}\, dt \quad = \int_0^{+\infty} f(u+T) e^{-s(u+T)}\, du \quad = e^{-sT} \int_0^{+\infty} f(u) e^{-su}\, du$$

$$t = u + T$$

$$F(s) = \int_0^T f(t) e^{-st}\, dt + e^{-sT} F(s)$$

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